

Module 12 - Navier - Stokes Equations:

Differential Form of the Conservation of Momentum

Equation:

see video 9.1

$$\frac{D\mathbf{u}}{Dt} = \mathbf{g} + \frac{\partial \mathbf{U}_{xx}}{\partial x} + \frac{\partial \mathbf{U}_{yx}}{\partial y} + \frac{\partial \mathbf{U}_{zx}}{\partial z}$$

$$\frac{D\mathbf{v}}{Dt} = \mathbf{g} + \frac{\partial \mathbf{U}_{xy}}{\partial x} + \frac{\partial \mathbf{U}_{yy}}{\partial y} + \frac{\partial \mathbf{U}_{zy}}{\partial z}$$

$$\frac{D\mathbf{w}}{Dt} = \mathbf{g} + \frac{\partial \mathbf{U}_{xz}}{\partial x} + \frac{\partial \mathbf{U}_{yz}}{\partial y} + \frac{\partial \mathbf{U}_{zz}}{\partial z}$$

where

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \frac{\partial \mathbf{u}}{\partial x} + \mathbf{v} \frac{\partial \mathbf{u}}{\partial y} + \mathbf{w} \frac{\partial \mathbf{u}}{\partial z}$$

see

video

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{u} \frac{\partial \mathbf{v}}{\partial x} + \mathbf{v} \frac{\partial \mathbf{v}}{\partial y} + \mathbf{w} \frac{\partial \mathbf{v}}{\partial z}$$

4.5

$$\frac{D\mathbf{w}}{Dt} = \frac{\partial \mathbf{w}}{\partial t} + \mathbf{u} \frac{\partial \mathbf{w}}{\partial x} + \mathbf{v} \frac{\partial \mathbf{w}}{\partial y} + \mathbf{w} \frac{\partial \mathbf{w}}{\partial z}$$

$$\underline{\text{inviscid}} \rightarrow \underline{\zeta_{xy} = \nu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)}$$

$$\nu = G$$

$$\underline{\zeta_{xy} = \zeta_{yx} = \zeta_{yz} = \zeta_{zy} = \zeta_{xz} = \zeta_{zx} = 0}$$

$$\underline{\zeta_{xx} = \zeta_{yy} = \zeta_{zz} = -p}$$

Viscous (Newtonian Fluids)

$$\underline{\zeta_{xy} = \zeta_{yx} = \nu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)}$$

$$\underline{\zeta_{yz} = \zeta_{zy} = \nu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)}$$

$$\underline{\zeta_{zx} = \zeta_{xz} = \nu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)}$$

$$\underline{\zeta_{xx} = -p + 2\nu \left(\frac{\partial u}{\partial x} \right)}$$

$$\underline{\zeta_{yy} = -p + 2\nu \left(\frac{\partial v}{\partial y} \right)}$$

$$\underline{\zeta_{zz} = -p + 2\nu \left(\frac{\partial w}{\partial z} \right)}$$

$$L_{zz} = -\rho + 2\rho \left(\frac{\partial \omega}{\partial z} \right)$$

$$\cancel{g} \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = - \frac{\partial p}{\partial x} + g g_x$$

~~x~~ dir a_x

$$+ \rho \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

~~y~~ dir $\cancel{g} \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) =$

$$- \frac{\partial p}{\partial y} + g g_y + \rho \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

~~z~~ dir $\cancel{g} \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) =$

$$- \frac{\partial p}{\partial z} + g g_z + \rho \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

NAVIER - STOKES EQUATIONS

In cylindrical polar coordinates:

r direction

$$g \left(\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z \frac{\partial V_r}{\partial z} \right)$$

$$= - \frac{\partial p}{\partial r} + g g_r + p \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V_r}{\partial r} \right) - \frac{V_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 V_r}{\partial \theta^2} \right.$$

$$\left. - \frac{2}{r^2} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial^2 V_r}{\partial z^2} \right]$$

θ direction

$$g \left(\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z \frac{\partial V_\theta}{\partial z} \right)$$

$$= - \frac{1}{r} \frac{\partial p}{\partial \theta} + g g_\theta + p \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V_\theta}{\partial r} \right) - \frac{V_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 V_\theta}{\partial \theta^2} \right]$$

$$+ \frac{2}{r^2} \frac{\partial V_r}{\partial \theta} + \frac{\partial^2 V_\theta}{\partial z^2}]$$

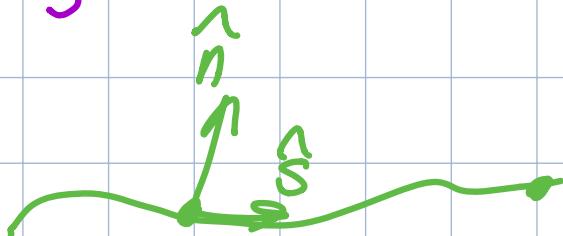
z-direction:

$$\rho \left(\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z} \right) =$$

$$- \frac{\partial P}{\partial z} + \rho g_z + p \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V_z}{\partial \theta^2} + \right.$$

$$\left. \frac{\partial^2 V_z}{\partial z^2} \right]$$

B.C's)

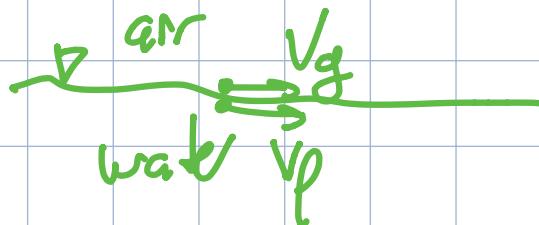


$V_s = U \rightarrow$ velocity of the solid surface

$$V_n = 0$$

$V_n = P \rightarrow$ penetration velocity

2) Free surface



$V_g = V_p$ at the interface

$\zeta_g = \zeta_p$ at the interface

Laminar, Turbulent, Transitional regimes

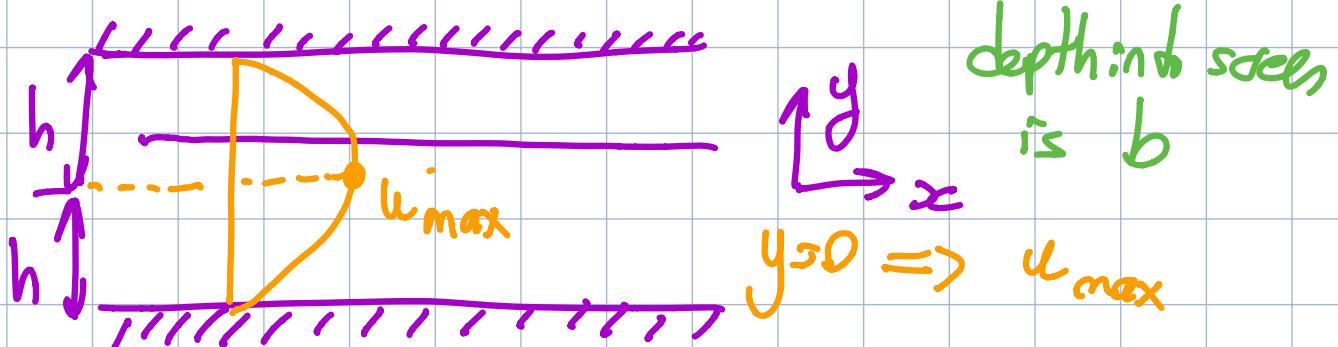
Entrance - Flow and Fully developed flow

See video ||. |

$$\xrightarrow{x}$$

$\frac{\partial p}{\partial x}$ is constant

Poiseuille Flow:



Steady, laminar, fully-developed flow between fixed parallel plates.

1) Steady ($\frac{\partial \dots}{\partial t} = 0$) 2) $v=0, w=0$ 3) $a = -a \hat{j}$

2-D

$$1) \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$$

~~$\frac{\partial u}{\partial x}$~~ ~~$\frac{\partial v}{\partial y}$~~ ~~$\frac{\partial w}{\partial z}$~~

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = 0$$

$$2) g \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = - \frac{\partial p}{\partial x} + g \cdot g$$

~~$\frac{\partial u}{\partial t}$~~ ~~$\frac{\partial u}{\partial x}$~~ ~~$\frac{\partial u}{\partial y}$~~ ~~$\frac{\partial u}{\partial z}$~~

C.O. Mass

$v=0$ $w=0$ $g \cdot g$

$$+ p \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

C.O. Mass

2-D

$$0 = - \frac{\partial p}{\partial x} + p \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{p} \frac{\partial p}{\partial x}$$

take integral w.r.t y Once

$$\frac{\partial u}{\partial y} = \frac{1}{p} \frac{\partial p}{\partial x} y + C_1$$

take integral w.r.t y once more

$$u(y) = \frac{1}{2\rho} \frac{\partial \rho}{\partial x} y^2 + C_1 y + C_2$$

B.C

$$y = +h \quad u = 0$$

$$y = -h \quad u = 0$$

$$0 = \frac{1}{2\rho} \frac{\partial \rho}{\partial x} h^2 + C_1 h + C_2$$

$$0 = \frac{1}{2\rho} \frac{\partial \rho}{\partial x} h^2 - C_1 h + C_2$$

$$C_1 = 0 \quad C_2 = -\frac{1}{2\rho} \frac{\partial \rho}{\partial x} h^2$$

$$u(y) = \frac{1}{2\rho} \frac{\partial \rho}{\partial x} y^2 - \frac{1}{2\rho} \frac{\partial \rho}{\partial x} h^2$$

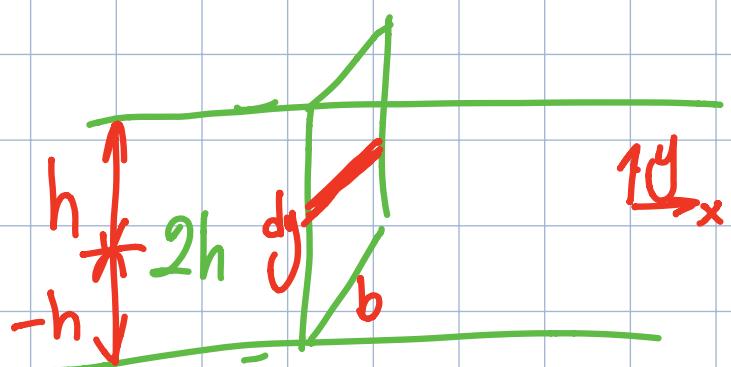
$$u(y) = \frac{1}{2\rho} \frac{\partial \rho}{\partial x} (y^2 - h^2)$$

$$y=0 \Rightarrow u_{\max} = \frac{1}{2p} \frac{\partial p}{\partial x} (-h^2)$$

$$\dot{m} = 100 \text{ kg/s} \quad q_e = 100 \text{ kJ/kg}$$

\bar{V} mean velocity

$$Q = \int_{-h}^h u \cdot b \cdot dy$$



$$Q = \int_{-h}^h \frac{1}{2p} \frac{\partial p}{\partial x} (y^2 - h^2) \cdot b \cdot dy$$

$$= \frac{1}{2p} \frac{\partial p}{\partial x} \cdot b \int_{-h}^h (y^2 - h^2) dy$$

$$= \frac{1}{2p} \frac{\partial p}{\partial x} \cdot b \left[\frac{y^3}{3} - h^2 y \right]_{-h}^h$$

$$= \frac{1}{2\rho} \frac{\partial P}{\partial x} b \left[\frac{h^3}{3} - h^3 \right] - \left[\frac{-h^3}{3} + h^3 \right]$$

$\frac{-2h^3}{3}$
 $\frac{2h^3}{3}$

$$= \frac{1}{2\rho} \frac{\partial P}{\partial x} b \left(\frac{-\frac{2}{3}h^3}{3} \right)$$

$$\underline{Q = -\frac{2h^3}{3\rho} \frac{\partial P}{\partial x} \cdot b}$$

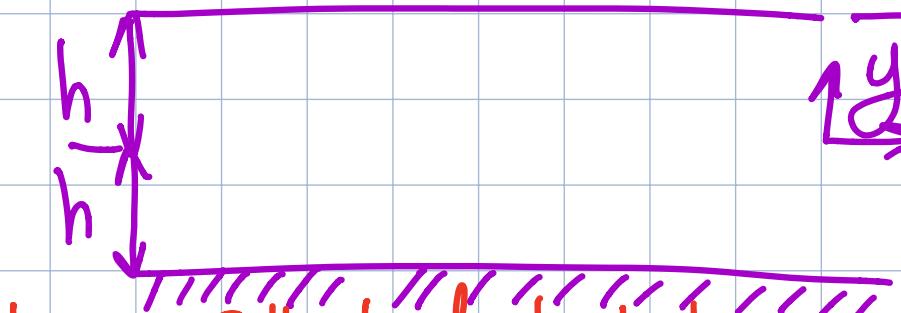
$$Q = \bar{V} \cdot A \Rightarrow -\frac{2h^2}{3\rho} \frac{\partial P}{\partial x} \cdot b = \bar{V} \cdot 2h \cdot b$$

$$\bar{V} = -\frac{h^2}{3\rho} \left(\frac{\partial P}{\partial x} \right)^3$$

$$U_{max} = -\frac{h^2}{2\rho} \left(\frac{\partial P}{\partial x} \right)$$

$$\boxed{U_{max} = \frac{3}{2} \bar{V}}$$

Couette Flow and Combined Couette-Poiseuille Flow:



$$\frac{\partial P}{\partial x} = 0 \quad \text{Couette Flow}$$

$$U_1 \rightarrow$$

1) 2-D

$$2) g = -g_y \hat{j}$$

3) steady

$$4) v=0, w=0$$

$\frac{\partial P}{\partial x}$ is non-zero Couette - Poiseuille Flow

$$1) u = C_1 y + C_2 \quad \text{for Couette Flow}$$

$$2) u = \frac{1}{2\rho} \frac{\partial P}{\partial x} y^2 + C_1 y + C_2 \quad \text{Couette - Poiseuille Flow}$$

$$\text{at } y=h \quad u = U$$

$$\text{at } y=-h \quad u = 0$$

$$1) u = U (1 + \frac{y}{h})$$

$$2) U = \frac{1}{2\rho} \frac{\partial P}{\partial x} (y^2 - h^2) + \frac{U}{2} \left(1 + \frac{y}{h} \right)$$

Poiseuille Flow

$$\rightarrow \frac{\partial f}{\partial x} = \frac{\partial P}{\partial x}$$

$$u=0$$

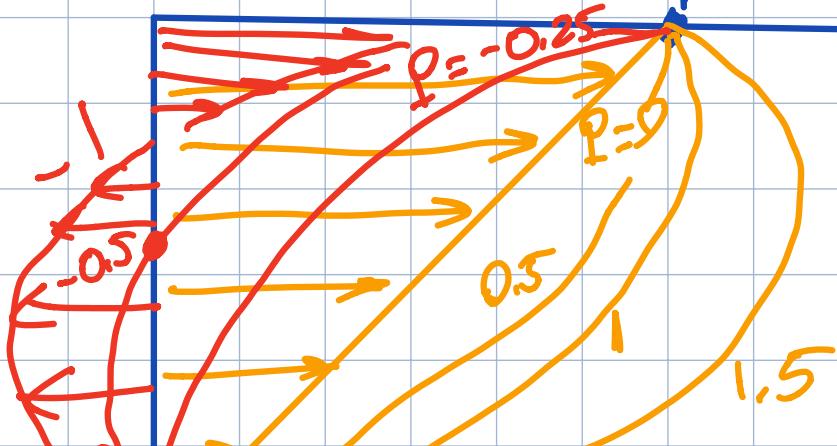
Couette Flow

$$u = U$$

$$+ \frac{\partial P}{\partial x} \approx 0$$

$$\frac{U}{U} = P \left(1 - \frac{y^2}{h^2} \right) + \frac{1}{2} \left(1 + \frac{y}{h} \right)$$

$$P = - \left(\frac{\partial P}{\partial x} \right) \frac{h^2}{2\rho \cdot U}$$

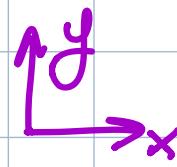
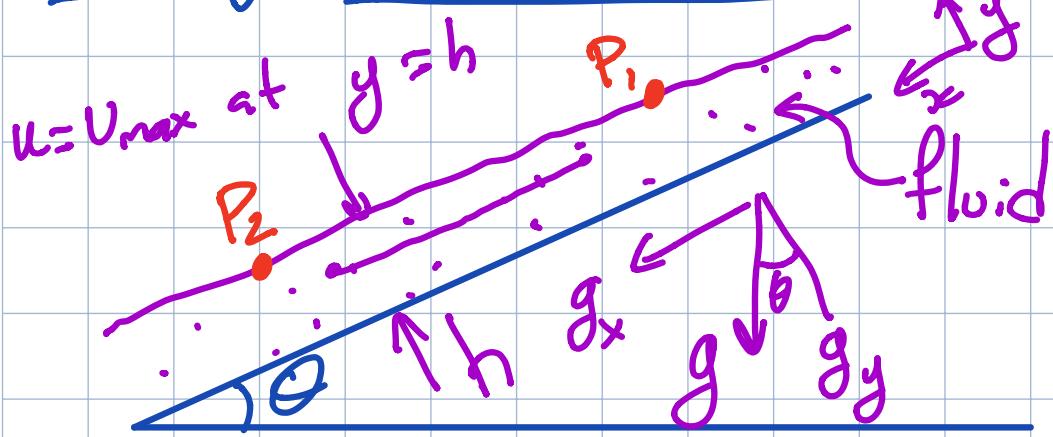


$$P = -0.25 \quad \gamma_{bottom} = 0$$

$$P < -0.25 \quad \text{backflow or}$$

Flow separation.

Gravity-Driven Liquid Film on an inclined surface:



$$P_1 = P_2 = 0$$

1) Laminar

$$u \neq 0, v = 0, w = 0$$

2) 2-D

$$3) \vec{g} = \underline{\underline{g}} \cdot \sin\theta \hat{i} - \underline{\underline{g}} \cdot \cos\theta \hat{j}$$

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} + \cancel{\frac{\partial w}{\partial z}} = 0$$

$$\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = 0$$

2) N-S in x dir $a_x = 0$

$$g \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)$$

Steady C.O.Mass $V=0$ $w=0$
 $= -\frac{\partial \phi}{\partial x} + g \sin \theta + p \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$
 C.O.Mass 2-D
 θ $g \sin \theta$

$$0 = g \cdot g \sin \theta + p \left[\frac{\partial^2 u}{\partial y^2} \right]$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{g \sin \theta}{p} \quad \text{take integral once}$$

$$\frac{\partial u}{\partial y} = -\frac{g \sin \theta}{p} y + C_1 \quad \text{take integral one more}$$

$$u(y) = -\frac{g \sin \theta}{2p} y^2 + C_1 y + C_2$$

at $y=0 \quad u=0$

$y=h \quad u_{max} \rightarrow \frac{du}{dy}=0$

$$T=0 \quad p \cdot \frac{du}{dy} = 0$$

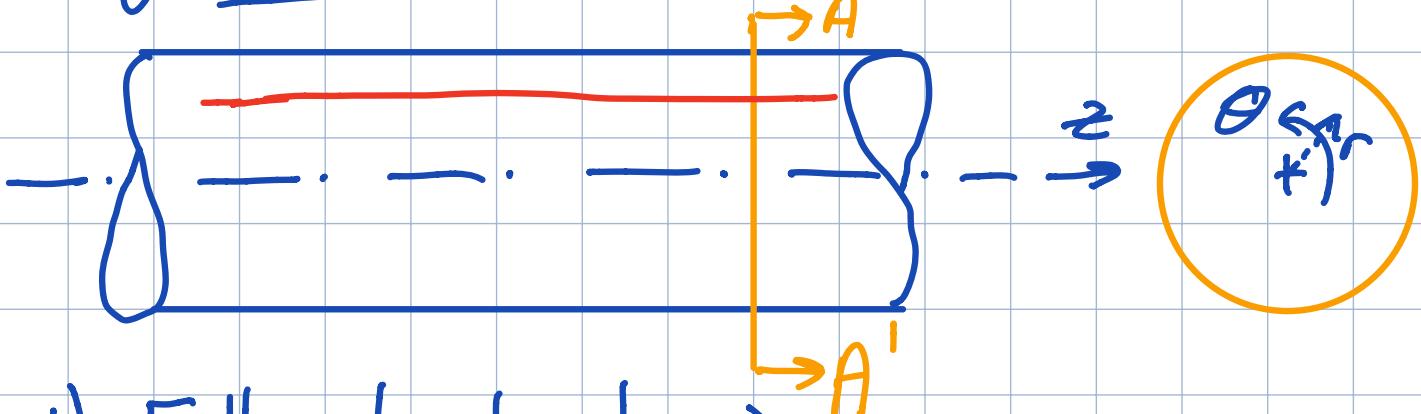
$$0 = -\frac{\rho g \sin \theta}{2\rho} (0)^2 + C_1(0) + C_2 \Rightarrow C_2 = 0$$

$$0 = -\frac{\rho g \sin \theta}{\rho} h + C_1$$

$$\Rightarrow C_1 = \frac{\rho g \sin \theta}{\rho} h$$

$$u(y) = \frac{\rho g \sin \theta}{2\rho} (2hy - y^2)$$

Hagen - Poiseuille Flow:



I) Fully-developed $\frac{\partial P}{\partial z} = \text{const.}$

2.) Laminar $V_r = 0 \quad V_\theta = 0 \quad V_z \neq 0$

3) V_z is independent of θ (axisymmetric)

C.O.Mass $\frac{1}{r} \frac{\partial(r)V_r}{\partial r} + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$

$\frac{\partial V_z}{\partial z} = 0$

N-S in z dir $a_z = 0$

$$\rho \left(\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z} \right)$$

$$= - \frac{\partial p}{\partial z} + \rho g_z + p \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V_z}{\partial \theta^2} \right]$$

$$g_z = 0$$

$$+ \frac{\partial^2 V_z}{\partial z^2}$$

$$\frac{\partial V_z}{\partial z} = 0$$

$$p \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V_z}{\partial r} \right) \right] = \frac{\partial p}{\partial z}$$

$$\frac{\partial}{\partial r} \left(r \frac{\partial V_z}{\partial r} \right) = \frac{r}{\rho} \frac{\partial \rho}{\partial z}$$

$$r \frac{\partial V_z}{\partial r} = \frac{1}{\rho} \frac{\partial \rho}{\partial z} \frac{r^2}{2} + C_1$$

divide both sides of the eqn by r

$$\frac{\partial V_z}{\partial r} = \frac{1}{2\rho} \frac{\partial \rho}{\partial z} r + \frac{C_1}{r}$$

take integral once more

$$V_z(r) = \frac{1}{4\rho} \frac{\partial \rho}{\partial z} r^2 + C_1 \ell_0(r) + C_2$$

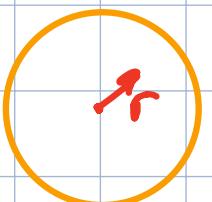
B.C's $V_z(R) = 0$ $C_1 = 0$

$$0 = \frac{R^2}{4\rho} \frac{\partial \rho}{\partial z} + C_2 \quad C_2 = -\frac{R^2}{4\rho} \frac{\partial \rho}{\partial z}$$

$$V_z(r) = -\frac{R^2}{4\mu} \frac{\partial p}{\partial z} \left[1 - \left(\frac{r}{R}\right)^2 \right]$$

$$V_{CL} = V_{max} = -\frac{R^2}{4\mu} \frac{\partial p}{\partial z}$$

$$V_z(r) = V_{max} \left[1 - \left(\frac{r}{R}\right)^2 \right]$$



$$Q = \int u dA = \int_0^R V_{max} \left[1 - \left(\frac{r}{R}\right)^2 \right] 2\pi r dr$$

$$Q = 2\pi V_{max} \int_0^R \left(r - \frac{r^3}{R^2} \right) dr$$

$$Q = 2\pi V_{max} \left[\frac{r^2}{2} - \frac{r^4}{4R^2} \right]_0^R$$

$$Q = 2\pi V_{max} \left[\frac{R^2}{2} - \frac{R^4}{4R^4} \right]_{(2)} - \frac{R^2}{4R^2} \frac{\partial P}{\partial z}$$

$$Q = 2\pi V_{max} \left[\frac{R^2}{K_2} \right]$$

$$Q = - \frac{R^4 \pi}{8P} \left[\frac{\partial P}{\partial z} \right]$$

$$\bar{V} = \frac{Q}{A} = - \frac{R^4 \pi}{8P} \left[\frac{\partial P}{\partial z} \right] \cancel{\pi R^2}$$

$$\bar{V} = - \frac{R^2}{8P} \left(\frac{\partial P}{\partial z} \right)$$

$$\frac{\partial P}{\partial z} = - \frac{\sqrt{8P}}{R^2}$$

$$V_{\max} = - \frac{R^2}{4\rho} \left(\frac{\partial \varphi}{\partial z} \right)$$

$$V = \frac{V_{\max}}{2}$$

$$\zeta_w = \rho \left(\frac{\partial V_z}{\partial r} \right) = \rho \cdot \frac{1}{2} \left[+ \frac{R^2}{4\rho} \left(\frac{\partial \varphi}{\partial z} \right) \left(+ \frac{2r}{R} \right) \left(+ \frac{2r}{R} \right) \right]$$

$$\zeta_w = \frac{R}{2} \frac{\partial \varphi}{\partial z}$$

$$= \frac{R}{2} \cdot \left(- \frac{\partial z}{V} \frac{\partial \varphi}{\partial r} \right)$$

$$\boxed{\zeta_w = - \frac{4\rho V}{R}}$$

$$C_f = - \frac{2 \zeta_w}{g V^2}$$

$$C_f = - 2 \cdot - \frac{4\rho V}{\rho} = \frac{8 \cdot \rho}{V}$$

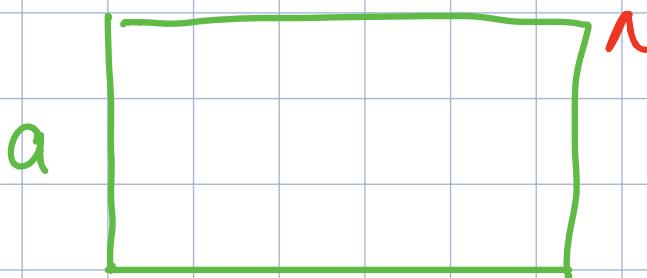
$$C_f = \frac{16}{\pi Re}$$

$$C_f = \frac{16}{Re}$$

$$Re = \frac{\rho V D}{\mu}$$

$$P_0 = C_f \cdot Re$$

↳ Poiseuille number



$$D_H = \frac{2ab}{a+b}$$