

Module 5: Conservation of Mass / Continuity Eqn.

Control Volume Analysis:

$$\text{R.TT} \quad \frac{DB}{Dt} = \frac{\partial}{\partial t} \iiint_{\text{mass}} g \cdot b \cdot dV + \iint_{\text{C.S.}} g \cdot b \vec{V} \cdot \hat{n} dA$$

B = mass

$$b = \frac{B}{\text{mass}} = \frac{\text{mass}}{\text{mass}} = 1$$

$$\frac{Dm}{Dt} = \frac{\partial}{\partial t} \iiint_{\text{C.V.}} g \cdot dV + \iint_{\text{C.S.}} g \cdot (\vec{V} \cdot \hat{n}) dA$$

Lagrangian
System (Control
Mass)

Eulerian. C.V.

$\frac{Dm}{Dt} = 0$ as m is constant for the system (C.M.)

$$\frac{\partial}{\partial t} \iiint_{\text{C.V.}} g dV + \iint_{\text{C.S.}} g (\vec{V} \cdot \hat{n}) dA = 0$$

time rate of change
of mass for the C.T

mass flux entering
or leaving the C.T

$$\vec{V} \cdot \hat{n} \rightarrow \begin{array}{ll} +V & \text{exit} \\ -V & \text{inlet} \\ 0 & \text{otherwise} \end{array}$$

$$\iint_{C.S.} g(\vec{V} \cdot \hat{n}) dA = \dot{m}_e - \dot{m}_i$$

mass flow rate

$$\dot{m}_e = \iint_{\text{exit}} g \cdot V \cdot dA$$

$$\dot{m}_i = - \iint_{\text{inlet}} g \cdot V \cdot dA$$

Special Cases of the C.O. Mass:

1) Steady flow: $\left(\frac{\partial \dots}{\partial t} = 0 \right)$

$$\frac{\partial}{\partial t} \iiint g \cdot dV = 0 \Rightarrow$$

c.t

$$\iint_{CS} g(\vec{V} \cdot \hat{n}) dA = 0$$

mass flow rate
 $\dot{m}_e - \dot{m}_i = 0$
 $\dot{m}_e = \dot{m}_i$

2) Steady + Constant Density:



$$g \iint_{CS} (\vec{V} \cdot \hat{n}) dA = 0$$

L.C.S
A. B

A=0 or
B=0 or
A=B=0

$$\iint_{CS} (\vec{V} \cdot \hat{n}) dA = 0 \quad \dot{Q}_e - \dot{Q}_i = 0$$

$\dot{Q}_e = \dot{Q}_i$

volumetric flow rate

3) Steady + Constant Density + Uniform flow





$$(\nabla \cdot \hat{n}) \iint dA = 0$$

C.S

A

$$\iint dx \rightarrow x$$

$$\iint dx dy =$$

$$-\sum_{\text{inlet}} V_{ni} A_i + \sum_{\text{exits}} V_{ne} A_e = 0$$

\dot{Q}_i \dot{Q}_e

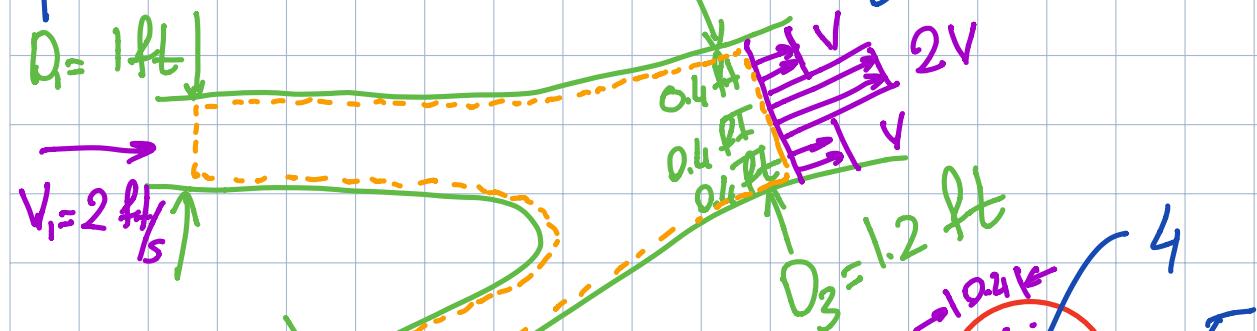
$$\iint dx dy$$

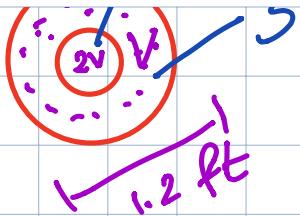
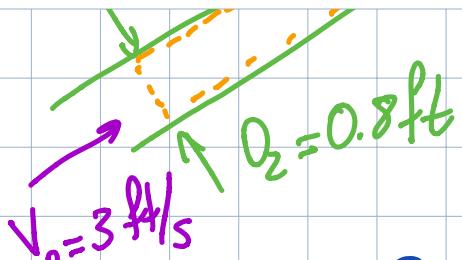
$$x \quad y = A$$

$$\dot{Q}_e = \dot{Q}_i$$

$$\sum_{\text{inlet}} V_{ni} A_i = \sum_{\text{exit}} V_{ne} A_e$$

Example: Two pipes merge into a single pipe. Before the flow is fully-developed, the velocity profile is as indicated in the figure. Find V





Assumptions:

- Steady
- Constant density
- Uniform flow for 1, 2, 4, 5

C.O. Mass

$$\sum_{\text{in}} V_i A_i = \sum_{\text{out}} V_o A_o$$

$$\sqrt{\pi \left[\frac{(1.2)^2 - 0.4^2}{4} \right]}$$

$$V_1 A_1 + V_2 A_2 = V_4 A_4 + V_5 A_5$$

$$\frac{V_1 A_1}{\frac{\pi (1)^2}{4}} + \frac{V_2 A_2}{\frac{\pi (0.8)^2}{4}} = \frac{V_4 A_4}{\frac{\pi (0.8)^2}{4}} + \frac{V_5 A_5}{\frac{\pi (0.4)^2}{4}}$$

$$2 \cdot \frac{(1)^2}{4} + 3 \cdot \frac{(0.8)^2}{4} = 2V \frac{(0.4)^2}{4} + V \left[\frac{(1.2)^2 - (0.4)^2}{4} \right]$$

$$0.25 + 0.16 = 0.04 + 0.32$$

$$0.41 = 0.04 + 0.32$$

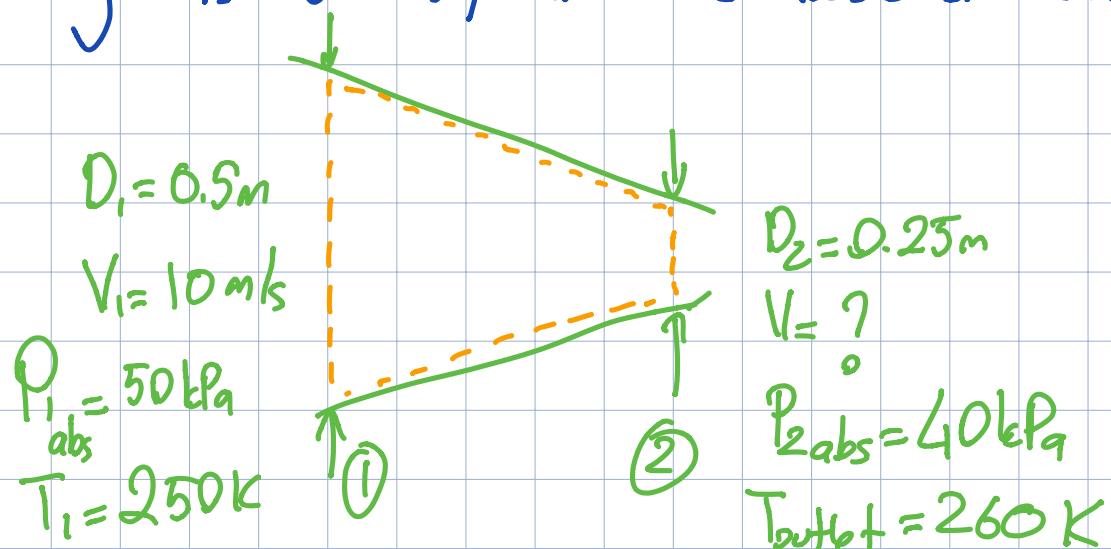
$$0.37 = 0.32$$

$$0.5 + 0.48 = 0.08V + 0.32V$$

$$0.98 = 0.4V$$

$$V = 2.45 \text{ ft/s}$$

Example: Air flows in a nozzle with dimensions indicated in the figure below. The pressures and temperatures at the nozzle inlet and outlet are also indicated in the figure below. If the nozzle inlet velocity is 10 m/s, what is the nozzle exit velocity.



Assumptions:

- steady
- Uniform

$$\sum_{\text{inlet}} \rho_i V_i A_i = \sum_{\text{exit}} \rho_e V_e A_e$$

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

$$V_2 = V_1 \cdot \frac{\rho_1}{\rho_2} \cdot \frac{A_1}{A_2}$$

ideal gas law

$$P_i = \rho_i \cdot R \cdot T_i \Rightarrow \rho \cancel{T} = \underline{m R T}$$

$$\underline{P_2} = \underline{S_2} \cdot \underline{R} \cdot T_2$$

$$\frac{S_1}{S_2} = \frac{P_1}{P_2} \cdot \frac{T_2}{T_1}$$

$$V_2 = V_1 \cdot \frac{P_1}{P_2} \cdot \frac{T_2}{T_1} \cdot \frac{A_1}{A_2}$$

$$= (10) \left(\frac{50 \text{ kPa}}{40 \text{ kPa}} \right) \left(\frac{260 \text{ K}}{250 \text{ K}} \right) \left(\frac{\pi (0.5)^2}{\pi (0.25)^2} \right)$$

$$= (10) (1.25) \left(\frac{260}{250} \right) \underline{4}$$

$$= 50 \cdot \frac{260}{250} = \boxed{52 \text{ m/s} = V_2}$$

for incorrect assumption of constant density

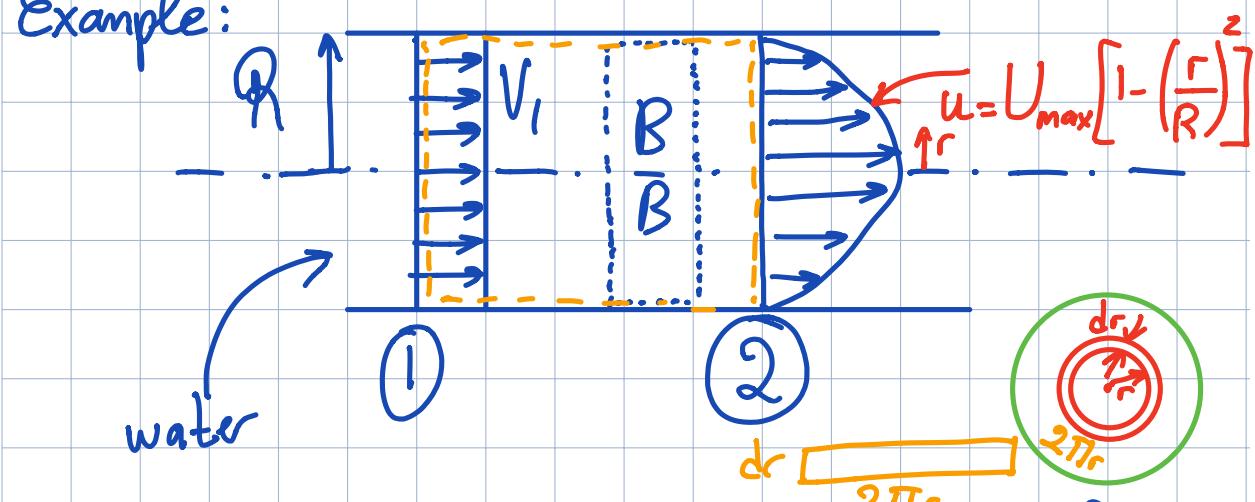
$$V_1 A_1 = V_2 A_2$$

$$(10) \left(\frac{\pi (0.5)^2}{4} \right) = V_2 \left[\frac{\pi (0.25)^2}{1} \right]$$

$$\underline{V_2 = 40 \text{ m/s}}$$

$$\frac{52 - 40}{52} \cdot 12$$

Example:



Obtain an expression for V_1 as a function of V_2

Assumptions: - steady; - constant density; uniform at (1)

$$\sum_{\text{inlet}} V_i \cdot A_i = \iint V_e dA_e$$

$$V_1 \cdot A_1 = \int_0^R U_{\max} \left[1 - \left(\frac{r}{R} \right)^2 \right] 2\pi r dr$$

$$V_1 \cdot \pi R^2 = U_{\max} \cdot 2\pi \int_0^R \left[1 - \left(\frac{r}{R} \right)^2 \right] r dr$$

$$V_1 R^2 = 2U_{\max} \int_0^R \left(r - \frac{r^3}{R^2} \right) dr$$

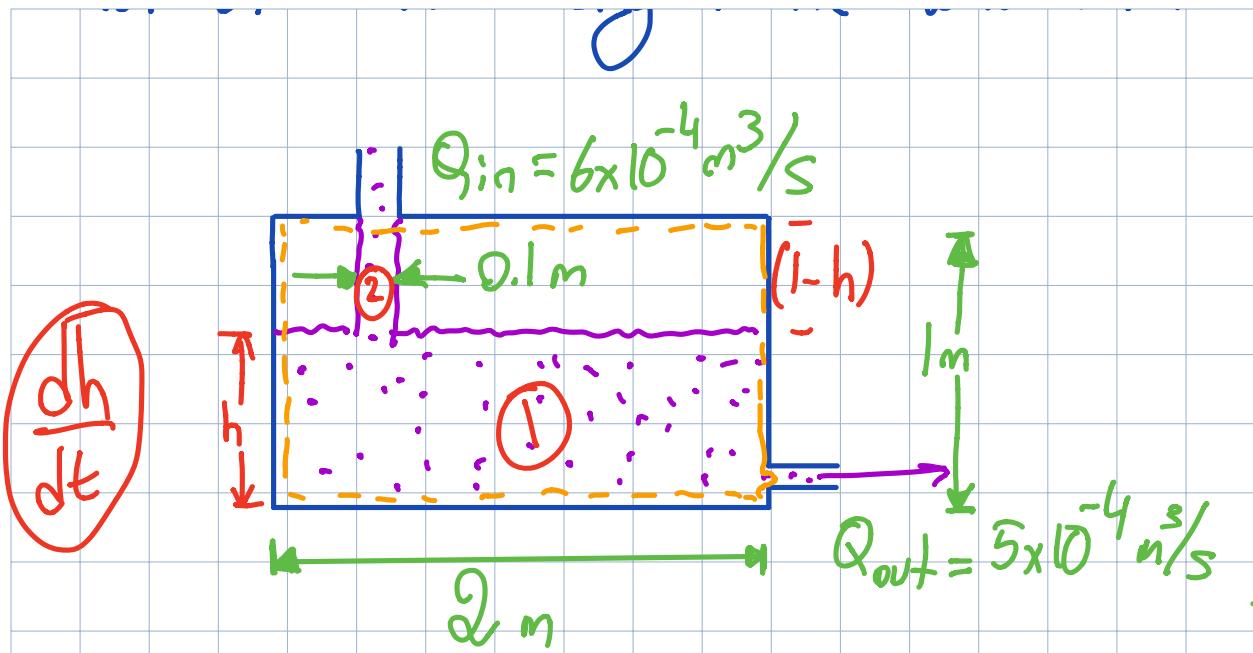
$$V_1 R^2 = 2U_{max} \left[\frac{r^2}{2} - \frac{r^4}{4R^2} \right]_0^R$$

$$V_1 R^2 = 2U_{max} \cdot \left[\frac{R^2}{2} - \frac{R^{4/2}}{4R^2} \right]$$

$$V_1 \cancel{R^2} = 2U_{max} \cdot \frac{R^2}{4}$$

$$V_1 = \frac{U_{max}}{2}$$

Example: 2m diameter water tank is being filled with volumetric flow rate of $6 \times 10^{-4} \text{ m}^3/\text{s}$, while the drain is removing $5 \times 10^{-4} \text{ m}^3/\text{s}$ volumetric flow rate of water. Find the velocity of the level of water rising in the water tank.



Assumptions - constant density - uniform flow

$$\frac{\partial}{\partial t} \iiint_{C.V} g \cdot dV + \iiint_{C.S} g \cdot (\vec{V} \cdot \hat{n}) dA = 0$$

$$g \frac{\partial}{\partial t} \iiint_{C.V} dV = g \cdot \frac{\partial V}{\partial t}$$

$$g \frac{\partial V}{\partial t} + \beta V e A_e - \sum_i g \cdot V_i A_i = 0$$

$5 \times 10^{-4} \text{ m}^3/\text{s}$ $6 \times 10^{-4} \text{ m}^3/\text{s}$

$\frac{dm}{dt}$

$$\frac{dm}{dt} = g \left(10^{-4} \right) m^3 / s = 0.1 \frac{\text{kg}}{\text{s}}$$

\downarrow
 kg/m^3

$$\frac{\partial V}{\partial t} = 10^{-4} \frac{\text{m}^3}{\text{s}}$$

$$\frac{\partial}{\partial t} (V_1 + V_2) = 10^{-4}$$

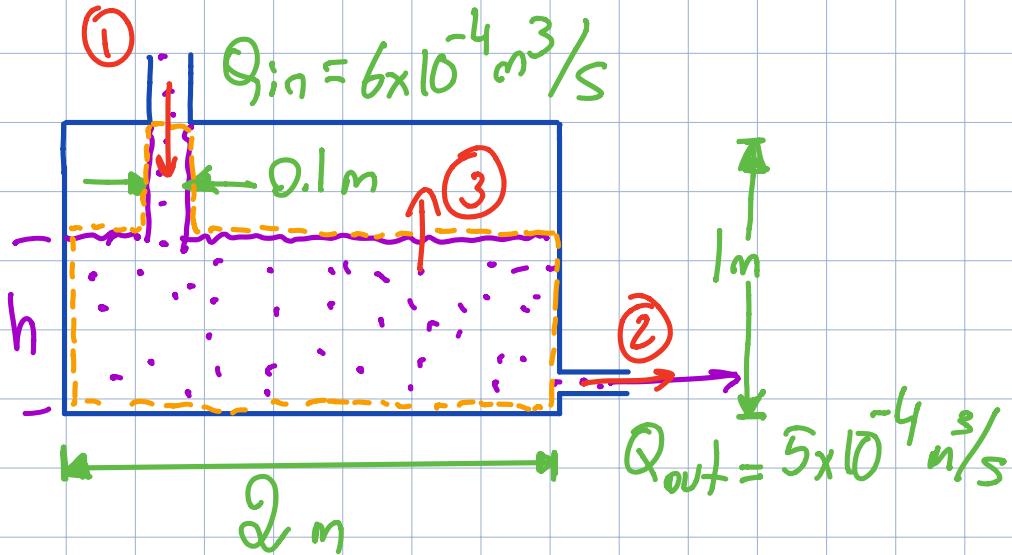
$$\frac{\partial}{\partial t} \left(\underbrace{\pi (1)^2 \cdot h}_{V_1} + \pi (0.05)^2 [1-h] \right)$$

$$\pi \frac{d}{dt} \left(\underbrace{h + (0.05)^2 - (0.05)^2 h}_{\rightarrow} \right) = 10^{-4}$$

$$\frac{d}{dt} \left[h - (2.5 \times 10^{-3}) h + (0.05)^2 \right] = 10^{-4}$$

$$\boxed{\frac{dh}{dt} = \frac{10^{-4}}{\pi} \frac{m}{s}}$$

Δt || → |



Assumption :- Steady - constant density - uniform flow

$$\sum_{\text{exists}} V_e A_e = \sum_{\text{Inlets}} V_i A_i;$$

$$V_2 A_2 + V_3 A_3 = V_1 A_1$$

$$5 \times 10^{-4} \quad \quad \quad 6 \times 10^{-4}$$

$$V_3 A_3 = 10^{-4}$$

$$\frac{dh}{dt} \left(\pi \left((1)^2 - (0.05)^2 \right) \right) = 10^{-4}$$

$$\pi \frac{dh}{dt} \left(1 - (0.05)^2 \right) = 0^{-4}$$

$$\boxed{\frac{dh}{dt} = \frac{10^{-4}}{\pi} \frac{m}{s}}$$