Differential form of Conservation of Momentum:

$$\sum \vec{F} = \frac{D\vec{L}}{Dt} \quad \text{momentum} \qquad = \frac{D}{Dt}(m\vec{V})$$

for a closed system $\Rightarrow$ $\sum \vec{F} = m \dfrac{D\vec{V}}{Dt}$

Consider a small fluid particle $(\forall) = \delta x \delta y \delta z$

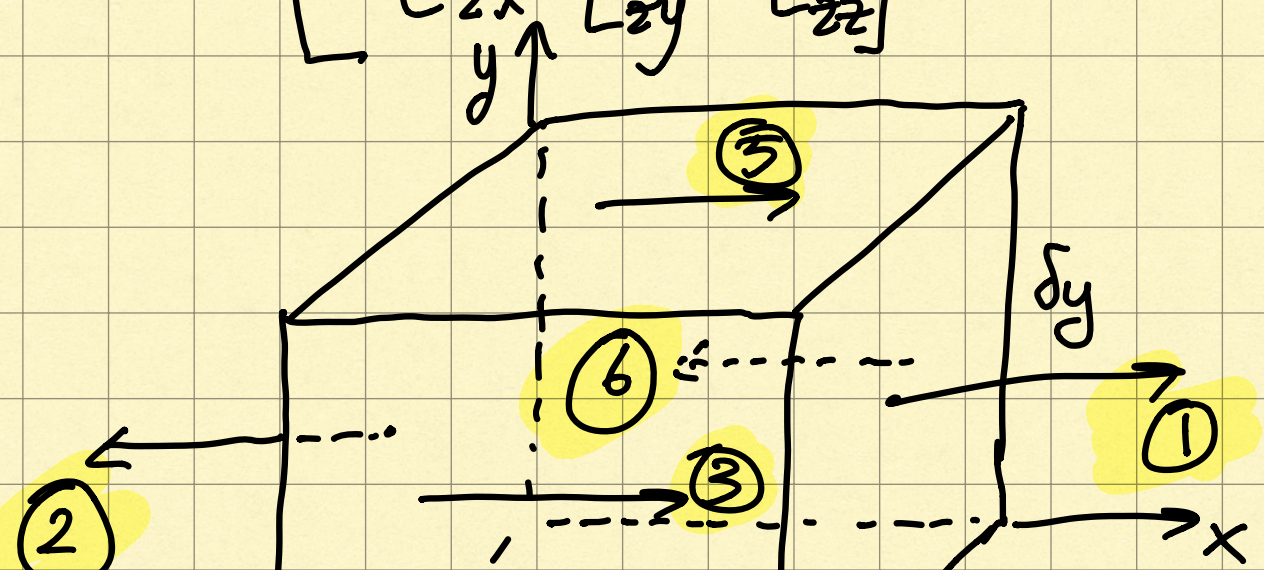
$$\sum \vec{F} = \sum \text{body forces} + \sum \text{surface forces}$$

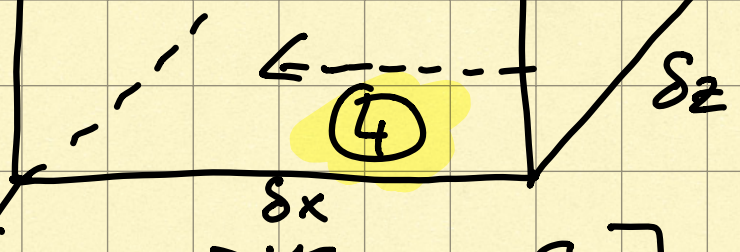
$$W = mg = \rho . \forall . g \qquad F = P . A$$

$$\sum \text{Body forces} = m\vec{g} = \rho \forall \left[\underline{g_x} \hat{i} + \underline{g_y} \hat{j} + \underline{g_z} \hat{k} \right]$$

$\sum$ surface forces $\rightarrow$ related to the stress tensor

$$\tau = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$





τ_{ij} → direction, i → normal

$$① \left[\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x} \cdot \frac{\delta x}{2} \right] \cdot \delta z \cdot \delta y$$

$$② \left[\tau_{xx} - \frac{\partial \tau_{xx}}{\partial x} \cdot \frac{\delta x}{2} \right] \cdot \delta y \delta z$$

$$③ \left[\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \cdot \frac{\delta z}{2} \right] \cdot \delta x \delta y$$

$$⑥ \left[\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z} \cdot \frac{\delta z}{2} \right] \cdot \delta x \delta y$$

$$⑤ \left[\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \cdot \frac{\delta y}{2} \right] \cdot \delta x \delta z$$

$$④ \left[\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y} \cdot \frac{\delta y}{2} \right] \cdot \delta x \delta z$$

$$\sum F_{surface} = \left[① - ② \right] + \left[③ - ⑥ \right] + \left[⑤ - ④ \right]$$

$$= \left[\frac{\partial \tau_{xx}}{\partial x} \frac{\delta x}{2} \right] \delta y \delta z + \left[\frac{\partial \tau_{xx}}{\partial x} \cdot \frac{\delta x}{2} \right] \delta y \delta z$$

$$= \cancel{2} \left[\frac{\partial \tau_{xx}}{\partial x} \cdot \frac{\delta x}{\cancel{2}} \right] \cdot \delta y \delta z \, \forall$$

$$= \frac{\partial \tau_{xx}}{\partial x} \cdot \forall$$

$$S_x = \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right] \cdot \forall$$

$$\underbrace{\rho \cdot \cancel{\forall}}_{m} \cdot \frac{Du}{Dt} = \underbrace{\rho \cdot \cancel{\forall} g_x}_{\text{Body force in x direction}} + \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right] \cancel{\forall}$$

$$\tau_{xx} \to -p \qquad \tau_{yy} \to 0 \qquad \tau_{zz} \to 0$$

$$\underline{\underline{x}} \quad \rho \frac{Du}{Dt} = \rho g_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\underline{\underline{y}} \quad \rho \frac{Dv}{Dt} = \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$\underline{\underline{z}} \quad \rho \frac{Dw}{Dt} = \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

where

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Special Case (inviscid flow) $\Rightarrow \mu = 0$

$$\tau = \underset{\substack{\uparrow \\ 0}}{\mu} \frac{du}{dy} \Rightarrow \text{no shear stresses.}$$

$$\tau_{xy} = \tau_{yx} = \tau_{xz} = \tau_{zx} = \tau_{yz} = \tau_{zy} = 0$$

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = -p$$

$$\rho \underbrace{\left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)}_{a_x} = \rho g_x - \frac{\partial p}{\partial x}$$

$$\rho \underbrace{\left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right)}_{a_y} = \rho g_y - \frac{\partial p}{\partial y}$$

$$\rho \underbrace{\left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)}_{a_z} = \rho g_z - \frac{\partial p}{\partial z}$$

EULER'S EQUATIONS

$\uparrow z$

$g_x = 0$ $\downarrow g$

$g_y = 0$

$g_z = -g$

$\Rightarrow$ 3 equations from Euler's eqn

1 equation from C.O. Mass

$(\underline{u}, \underline{v}, \underline{w}, \underline{p})$ unknown 4 unknowns, 4 equations.

Bernoulli's equation for potential or irrotational flow:

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

only applicable for any 2 points on a streamline

$$\Rightarrow \quad \frac{\partial u}{\partial y} \overset{?}{=} \frac{\partial v}{\partial x} \qquad \frac{\partial u}{\partial z} \overset{?}{=} \frac{\partial w}{\partial x} \qquad \frac{\partial v}{\partial z} \overset{?}{=} \frac{\partial w}{\partial y}$$

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

this equation is applicable between any 2 points for irrotational or potential flow

$$\underbrace{\frac{P_1}{\gamma}}_{\text{pressure head}} + \underbrace{\frac{V_1^2}{2g}}_{\text{velocity head}} + \underbrace{z_1}_{\text{elevation head}} = \frac{P_2}{\gamma_2} + \frac{V_2^2}{2g} + z_2$$

Euler's equation → any case

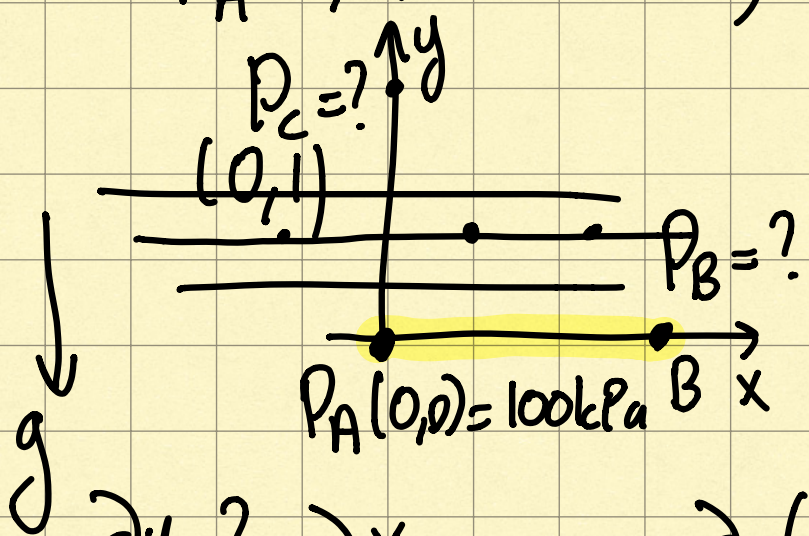
Bernoulli's equation → for two points on a streamline

↳ for any two points for potential flow

Example: A flow field for a steady, inviscid and incompressible fluid (water) is $u = y^2 + 2y + 5$

$v, w = 0$

If $P_A(0,0) = 100\ kPa$, find $P_B(1,0)$ and $P_C(0,1)$



$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}}^{\,0} + \cancel{\frac{\partial w}{\partial z}}^{\,0} = 0$$

$$\frac{\partial}{\partial x}\left(y^2 + 2y + 5\right) \overset{?}{=} 0 \checkmark$$

$$\frac{\partial u}{\partial y} \overset{?}{=} \frac{\partial v}{\partial x} \Rightarrow \frac{\partial}{\partial y}\left(y^2 + 2y + 5\right) \overset{?}{=} \frac{\partial}{\partial x}(0)$$

$$2y + 2 \neq 0 \quad \times$$

not irrotational → rotational

A and B are on the same streamline.

$$\frac{P_A}{\rho} + \cancel{\frac{V_A^2}{2}} + \cancel{g z_A} = \frac{P_B}{\rho} + \cancel{\frac{V_B^2}{2}} + \cancel{g z_B}$$

$V_A(0,0)$ $\quad u_A(0,0) = (0)^2 + 2(0) + 5 = 5$

$v_A(0,0) = 0$

$$V_A = \sqrt{(5)^2 + (0)^2} = 5$$

$V_B(1,0)$ $\quad u_B(1,0) = (0)^2 + 2(0) + 5 = 5$

$v_B(1,0) = 0 \quad w_B = 0$

$$V_B = \sqrt{(5)^2 + (0)^2 + (0)^2} = 5$$

$$\frac{P_A}{\cancel{\rho}} = \frac{P_B}{\cancel{\rho}} \Longrightarrow \boxed{P_A = P_B = 100\,kPa}$$

$$\rho \underbrace{\left(\cancel{\frac{\partial v}{\partial t}} + u \cancel{\frac{\partial v}{\partial x}} + v \cancel{\frac{\partial v}{\partial y}} + w \cancel{\frac{\partial v}{\partial z}} \right)}_{a_y = 0} = \rho g_y - \frac{\partial p}{\partial y}$$

$v = 0$

$$\frac{\partial p}{\partial y} = \rho g_y \longrightarrow \frac{\partial p}{\partial y} = 0$$

p is constant

$P_C = 100\,kPa$

(1000) $\quad (-9.81)$

$$\frac{dp}{dy} \leftarrow \frac{\partial p}{\partial y} = -9810 \Longrightarrow p(y) = -9810y + C$$

$p(0,0) = 100\,kPa$

$$100{,}000 = -9810(0) + C \Longrightarrow C = 100{,}000$$

$$p(y) = -9810(y) + 100{,}000$$

$$P_c(0,1) \Rightarrow -9810(1) + 100{,}000$$

$$P_c(0,1) = 90{,}190 \text{ Pa} \Rightarrow \underline{90.2 \text{ kPa}}$$

The streamfunction for a inviscid, incompressible flow is:

$$\psi = x^2 + y^2 + 2$$

a) is the conservation of mass satisfied (is this physically possible flow?)

b) if so, find the velocity potential

c) what is the pressure gradient in the horizontal x direction at the origin (0,0)

a) $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$

$$u = \frac{\partial}{\partial y}\left(x^2 + y^2 + 2\right) = 2y$$

$$v = -\frac{\partial}{\partial x}\left(x^2 + y^2 + 2\right) = -2x$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}}^{\,0} \overset{?}{=} 0 \qquad \frac{\partial}{\partial x}(2y) + \frac{\partial}{\partial y}(-2x) \overset{?}{=} 0$$

$0+0=0$ ✓

b) $\dfrac{\partial u}{\partial y} \overset{?}{=} \dfrac{\partial v}{\partial x}$ $\qquad \dfrac{\partial}{\partial y}(2y) \overset{?}{=} \dfrac{\partial}{\partial x}(-2x)$

$2 \neq -2$ X

not irrotational, velocity potential does not exist.

c) $\dfrac{\partial p}{\partial x}$

$$\rho\left(\cancel{\frac{\partial u}{\partial t}}^{0} + \cancel{u\frac{\partial u}{\partial x}} + \underset{-2x}{v}\underset{2}{\frac{\partial u}{\partial y}} + \cancel{w\frac{\partial u}{\partial z}}\right) = \cancel{\rho g_x}^{0} - \frac{\partial p}{\partial x}$$

$u=2y \qquad v=-2x$

$-4x = -\dfrac{\partial p}{\partial x} \Rightarrow \dfrac{\partial p}{\partial x} = 4x \quad (0,0)$

$$\boxed{\frac{\partial p}{\partial x} = 0}$$

Consider the following 2-D velocity potential

$$\phi = x^2 - y^2 + 3xy + 5$$

a) for an incompressible flow, is the conservation of mass satisfied?

b) what is the pressure difference between points A(0,0) and B(1,1)

a) $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \cancelto{0}{\frac{\partial^2 \phi}{\partial z^2}} \overset{?}{=} 0$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(x^2 - y^2 + 3xy + 5\right) = \frac{\partial}{\partial x}\left(2x + 3y\right) = 2$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2}{\partial y^2}\left(x^2 - y^2 + 3xy + 5\right) = \frac{\partial}{\partial y}\left(-2y + 3x\right) = -2$$

$$(\cancel{2}) + (-\cancel{2}) + (0) \overset{?}{=} 0 \checkmark$$

yes, conservation of mass is satisfied.

b) $u = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(x^2 - y^2 + 3xy + 5\right) = 2x + 3y$

$v = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(x^2 - y^2 + 3xy + 5\right) = -2y + 3x$

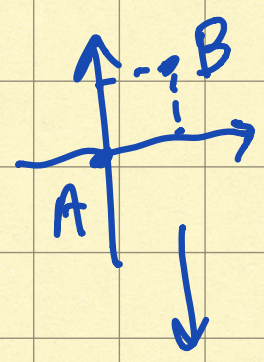
$\Delta p = P_B - P_A = P_{(1,1)} - P_{(0,0)}$

$u(0,0) = 2(0) + 3(0) = 0 \qquad v(0,0) = -2(0) + 3(0) = 0$

$V(0,0) = 0$

$u(1,1) = 2(1) + 3(1) = 5 \qquad v(1,1) = -2(1) + 3(1) = 1$

$V(1,1) = \sqrt{(5)^2 + (1)^2} = \sqrt{26}$



$$\frac{P_A}{\rho} + \frac{V_A^2}{2} + g z_A = \frac{P_B}{\rho} + \frac{V_B^2}{2} + g z_B$$

$$\frac{P_B - P_A}{\rho} = \underbrace{\frac{V_A^2 - V_B^2}{2}}_{0} + g\underbrace{(z_A - z_B)}_{(-1)}$$

$V_B^2 \to \sqrt{26}$

$$P_B - P_A = \left[(-13) + (9.81)(-1)\right] \cdot (1.23)$$

$$\boxed{P_B - P_A = -28 \text{ Pa}}$$

with g in -y direction

$$\boxed{P_B - P_A = -22.8}$$

with g aligned with z direction